Binary operations (See D+F 1.1)

Algebra is essentially the study of sets equipped with various "binary operations":

Examples:

Some familiar binary operations:

1.) $+, \cdot, -$ on $\mathbb{R}$ (or $\mathbb{Q}$ or $\mathbb{Z}$)

2.) $\div$ on $\mathbb{R}-\{0\}$ or $\mathbb{Q}-\{0\}$ (why not $\mathbb{Z}-\{0\}$?)

3.) Addition "mod 3".

i.e. $\{0,1,2\}$, where $a+b$ = remainder when dividing by 3.

Addition table:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

4.) min is a binary op. on $\mathbb{R}$, defined

$$\min(a,b) = \begin{cases} a & \text{if } a \le b \\ b & \text{if } b \le a \end{cases}$$

We can also make up our own binary ops:

5.) Define $*$ on $\mathbb{R}$ to be $a * b = 2a + 5b$.

So $3 * (-7) = 2 \cdot 3 + 5(-7) = -29$

and $(-7) * 3 = 2(-7) + 5 \cdot 3 = 1$

Def: If A is a set, then a <u>binary operation</u> $*$ on A is a function

$$* : A \times A \longrightarrow A.$$

Denote $*(a, b)$ by $a * b$.

Just like the binary operations w/ which we're already familiar, we mostly care about ones that satisfy nice properties.

Def: $*$ on A is <u>commutative</u> if $a * b = b * a \quad \forall \ a, b \in A.$

Ex: Let $A = \{ f : \mathbb{R} \to \mathbb{R} \}$ = the set of functions from $\mathbb{R}$ to $\mathbb{R}$.

$+$, $\cdot$, and $\circ$ are all binary operations on A.

$f + g$ is the function $(f+g)(x) = f(x) + g(x)$

$f \cdot g$ is defined $(fg)(x) = (f(x))(g(x))$

$f \circ g$ is defined $(f \circ g)(x) = f(g(x))$.

In this case, $+$ and $\cdot$ are both commutative, but if $f(x) = x^2$ and $g(x) = x + 1$, then

$(f \circ g)(x) = (x+1)^2$ and $(g \circ f)(x) = x^2+1$ so $f \circ g \neq g \circ f$, so $\circ$ is not commutative.

Def: $*$ is <u>associative</u> if $\forall\ a, b, c \in A$,

$$(a * b) * c = a * (b * c)$$

Ex: $\cdot$ on $\mathbb{R}$ is associative: $(ab)c = a(bc)$. However, $\div$ on $\mathbb{R} - \{0\}$ is not associative: $(2/1)/2 = 2/2 = 1$, but $2/(1/2) = 4$.